EECE 301
Signals & Systems
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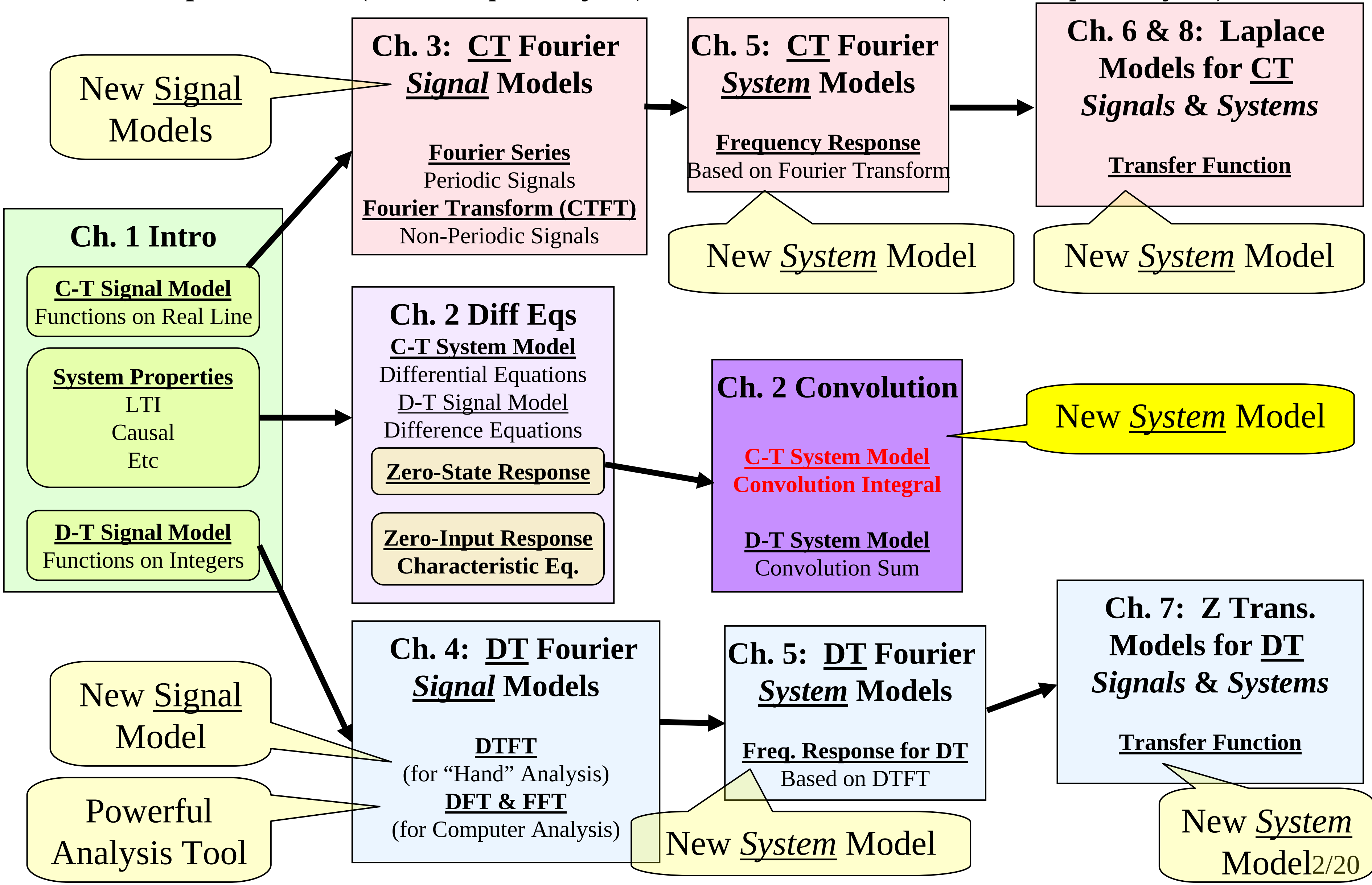
Note Set #11

- C-T Systems: "Computing" Convolution
- Reading Assignment: Section 2.6 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



C-T convolution properties

Many of these are the same as for DT convolution.

We only discuss the new ones here.

See the next slide for the others

derivative

1. **Derivative Property:**

$$\frac{d}{dt}[x(t)*v(t)] = \dot{x}(t)*v(t)$$
$$= x(t)*\dot{v}(t)$$

2. **Integration Property** Let $y(t) = x(t)*h(t)$, then

$$\int_{-\infty}^{t} y(\lambda)d\lambda = \left[\int_{-\infty}^{t} x(\lambda)d\lambda\right]*h(t) = x(t)*\left[\int_{-\infty}^{t} h(\lambda)d\lambda\right]$$

The properties of convolution help perform analysis and design tasks that involve convolution. For example, the associative property says that (in theory) we can interchange to order of two linear systems… in practice, before we can switch the order we need to check what impact that might have on the physical interface conditions.

Convolution Properties

These are things you can exploit to make it easier to solve convolution problems

1. Commutativity $$x(t) * h(t) = h(t) * x(t)$$

$\Rightarrow$ You can choose which signal to "flip"

2. Associativity $$x(t) * (v(t) * w(t)) = (x(t) * v(t)) * w(t)$$

$\Rightarrow$ Can change order $\rightarrow$ sometimes one order is easier than another

3. Distributivity $$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

$\Rightarrow$ may be easier to split complicated system $h[n]$ into sum of simple ones

OR

$\Rightarrow$ we can split complicated input into sum of simple ones

(nothing more than "linearity")

4. Convolution with impulses

$$x(t) * \delta(t - \tau) = x(t - \tau)$$

“Computing” CT Convolution

-For D-T systems, convolution is something we do for analysis and for implementation (either via H/W or S/W).

-For C-T systems, we do convolution for analysis…

nature does convolution for implementation.

If we are **analyzing** a given system (e.g., a circuit) we may need to compute a convolution to determine how it behaves in response to various different input signals

If we are **designing** a system (e.g., a circuit) we may need to be able to visualize how convolution works in order to choose the correct type of system impulse response to make the system work the way we want it to.

We’ll learn how to perform “**Graphical Convolution**,” which is nothing more than steps that help you use graphical insight to evaluate the convolution integral.

Steps for Graphical Convolution $x(t)*h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Note: I use τ for what the book uses λ... It is not a big deal as they are just dummy variables!!!

1. **Re-Write the signals as functions of τ**: $x(\tau)$ and $h(\tau)$

2. **Flip** just one of the signals around $t = 0$ to get either $x(-\tau)$ or $h(-\tau)$
   a. It is usually best to flip the signal with shorter duration
   b. For notational purposes here: we'll flip $h(\tau)$ to get $h(-\tau)$

3. **Find Edges** of the flipped signal
   a. Find the left-hand-edge τ-value of $h(-\tau)$: **call it $\tau_{L,0}$**
   b. Find the right-hand-edge τ-value of $h(-\tau)$: **call it $\tau_{R,0}$**

4. **Shift** $h(-\tau)$ by an arbitrary value of t to get $h(t - \tau)$ and **get its edges**
   a. Find the left-hand-edge τ-value of $h(t - \tau)$ as a function of t: **call it $\tau_{L,t}$**
      - **Important:** It will *always* be… $\tau_{L,t} = t + \tau_{L,0}$

   b. Find the right-hand-edge τ-value of $h(t - \tau)$ as a function of t: **call it $\tau_{R,t}$**
      - **Important:** It will *always* be… $\tau_{R,t} = t + \tau_{R,0}$

Note: If the signal you flipped is NOT finite duration,
one or both of $\tau_{L,t}$ and $\tau_{R,t}$ will be infinite ($\tau_{L,t} = -\infty$ and/or $\tau_{R,t} = \infty$)

Steps Continued

5. <u>Find Regions of τ-Overlap</u>

a. What you are trying to do here is find intervals of t over which the product $x(\tau)\, h(t - \tau)$ has a single mathematical form in terms of τ

b. In each region find: Interval of t that makes the identified overlap happen

c. Working examples is the best way to learn how this is done

<u>Tips</u>: Regions should be contiguous with no gaps!!!
Don't worry about $<$ vs. $\leq$ etc.

6. For Each Region: **<u>Form the Product $x(\tau)\, h(t - \tau)$ and Integrate</u>**

a. <u>Form product</u> $x(\tau)\, h(t - \tau)$

b. <u>Find the Limits of Integration</u> by finding the interval of τ over which the product is nonzero

   i. Found by seeing where the edges of $x(\tau)$ and $h(t - \tau)$ lie

   ii. Recall that the edges of $h(t - \tau)$ are $\tau_{L,t}$ and $\tau_{R,t}$, which often depend on the value of t

      • So… the limits of integration <u>may</u> depend on t

c. <u>Integrate the product</u> $x(\tau)\, h(t - \tau)$ over the limits found in 6b

   i. The result is generally a function of t, but is only valid for the interval of t found for the current region

   ii. Think of the result as a "time-section" of the output $y(t)$

Steps Continued

7. **<u>"Assemble" the output</u>** from the output time-sections for all the regions

a. Note: you do NOT add the sections together

b. You define the output "piecewise"

c. Finally, if possible, look for a way to write the output in a simpler form

Example: Graphically Convolve Two Signals

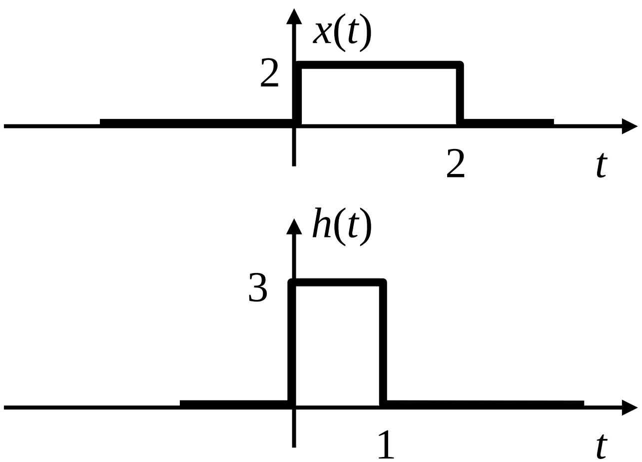
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

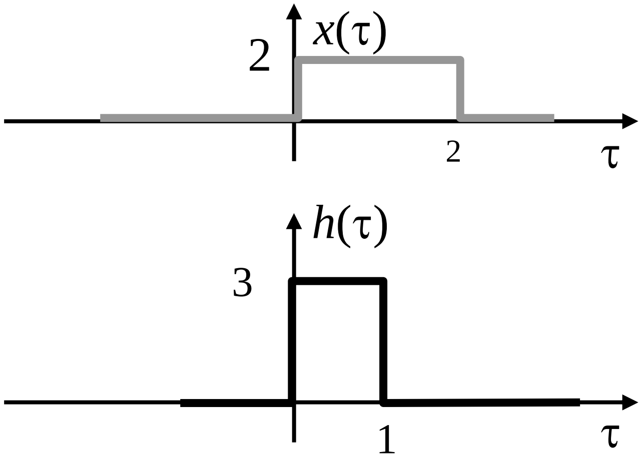
By "Properties of Convolution"… these two forms are Equal

<u>This</u> is why we can flip <u>either</u> signal

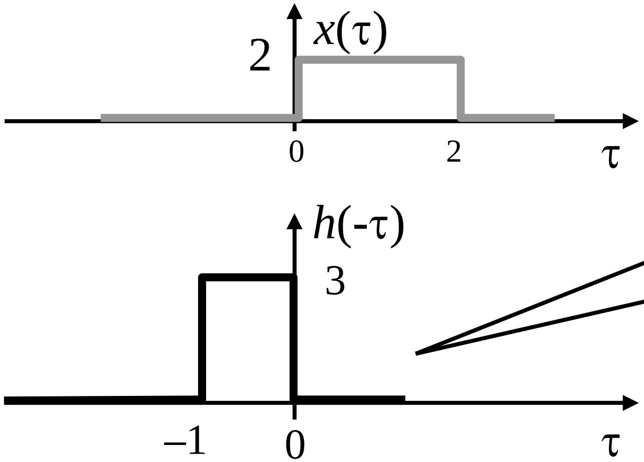
Convolve these two signals:



Step #1: Write as Function of τ

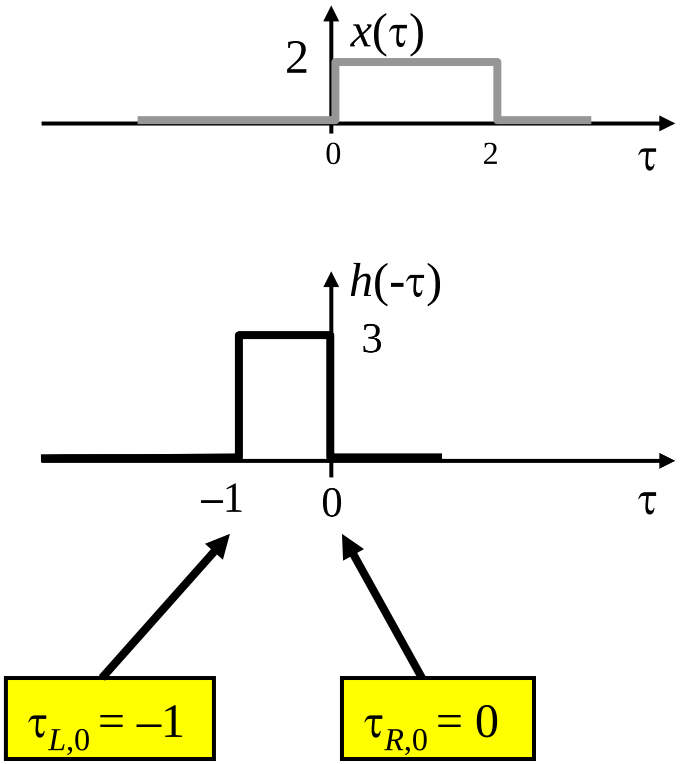


Step #2: Flip *h*(τ) to get *h*(-τ)



Usually Easier to Flip the Shorter Signal

Step #3: Find Edges of Flipped Signal



Motivating Step #4: Shift by t to get $h(t-\tau)$ & Its Edges

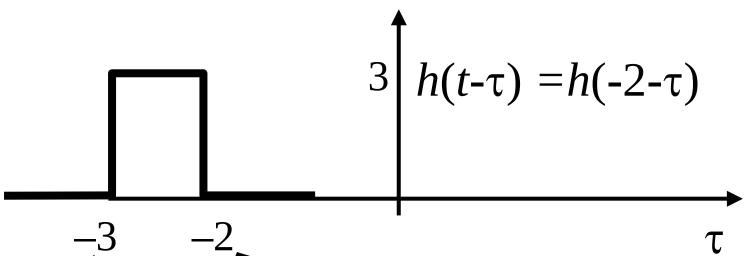
Just looking at 2 "arbitrary" t values

In Each Case We Get

$$\tau_{L,t} = t + \tau_{L,0}$$

$$\tau_{R,t} = t + \tau_{R,0}$$

For t = -2



$\tau_{L,t} = \boldsymbol{t} + \tau_{L,0}$

$\tau_{L,t} = \boldsymbol{t - 1}$

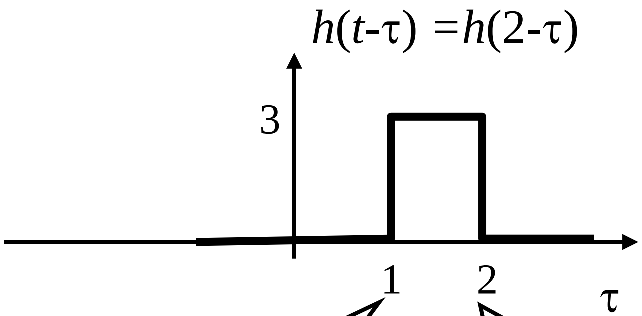
$\tau_{L,-2} = \mathbf{-2 - 1}$

$\tau_{R,t} = \boldsymbol{t} + \tau_{R,0}$

$\tau_{R,t} = \boldsymbol{t} + \mathbf{0}$

$\tau_{R,-2} = -2 + \mathbf{0}$

For t = 2



$\tau_{L,t} = \boldsymbol{t} + \tau_{L,0}$

$\tau_{L,t} = \boldsymbol{t - 1}$

$\tau_{L,2} = \mathbf{2 - 1}$

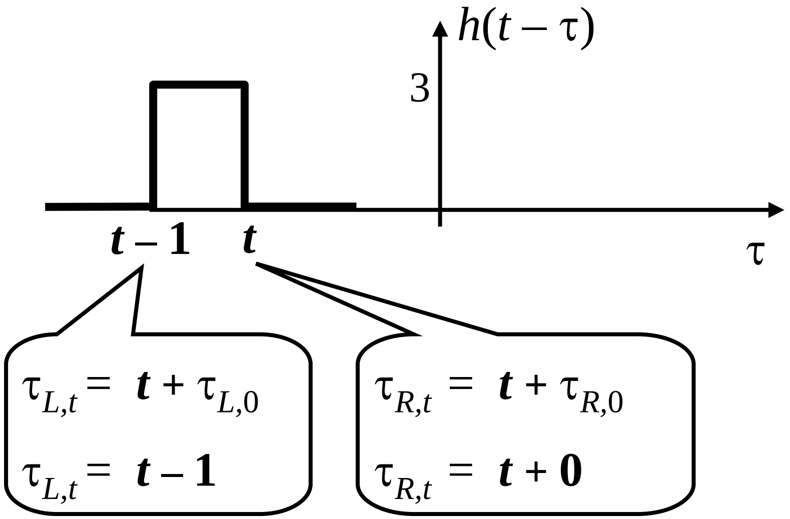
$\tau_{R,t} = \boldsymbol{t} + \tau_{R,0}$

$\tau_{R,t} = \boldsymbol{t} + \mathbf{0}$

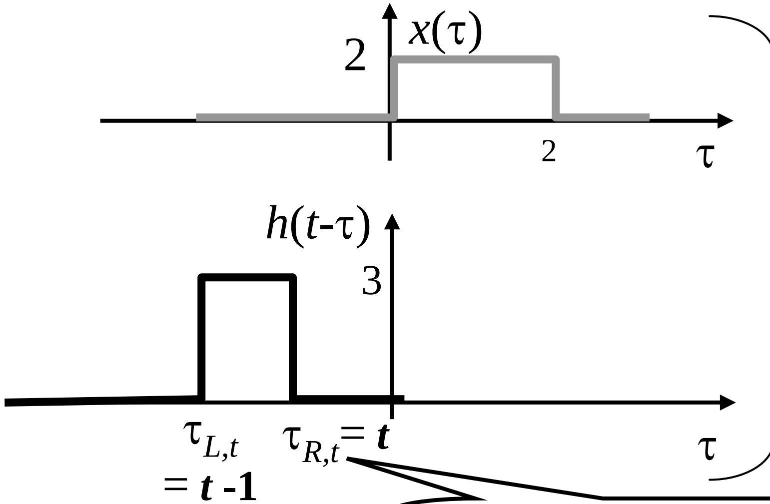
$\tau_{R,2} = 2 + \mathbf{0}$

Doing Step #4: Shift by *t* to get *h*(*t*-τ) & Its Edges

For *Arbitrary* Shift by *t*



Step #5: Find Regions of τ-Overlap

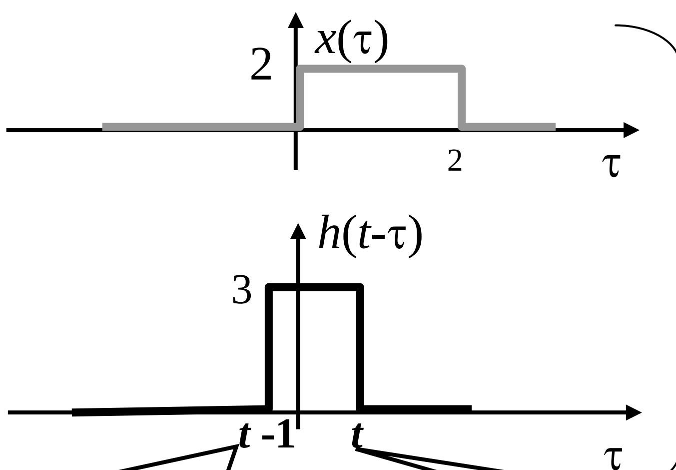


Want $\tau_{R,t} < 0$ ➔ $t < \mathbf{0}$

Region I

No τ-Overlap

$\boldsymbol{t < 0}$



Want $\tau_{L,t} \le 0$ ➔ $t-1 \le \mathbf{0}$ ➔ $t \le \mathbf{1}$

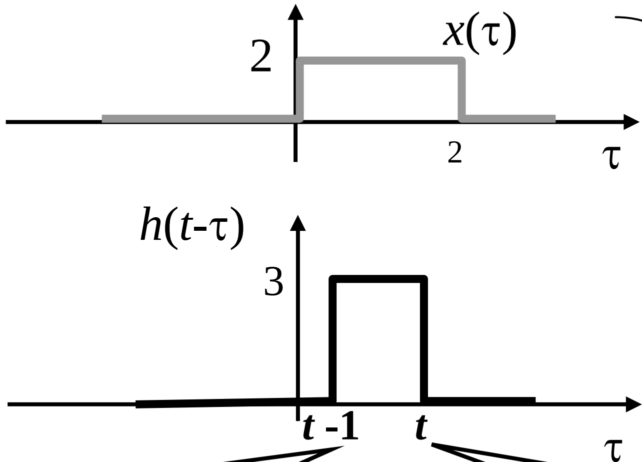
Want $\tau_{R,t} \ge 0$ ➔ $t \ge \mathbf{0}$

Region II

Partial τ-Overlap

$\boldsymbol{0 \le t \le 1}$

Step #5 (Continued): Find Regions of τ-Overlap



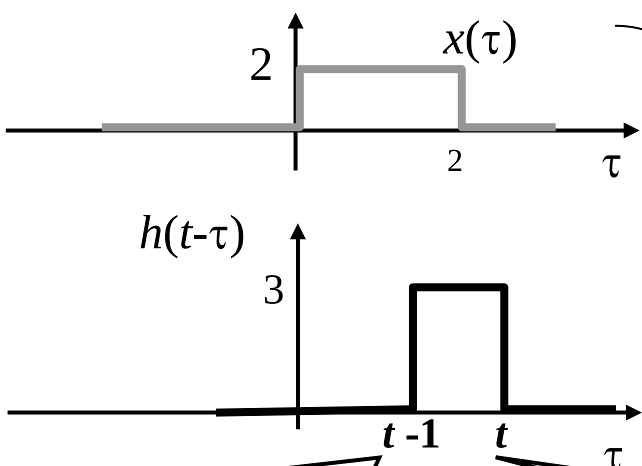
Region III

Total τ-Overlap

$\mathbf{1 < t \le 2}$

Want $\tau_{L,t} > 0$ ➔ t-1 > **0** ➔ $\boldsymbol{t > 1}$

Want $\tau_{R,t} \le 2$ ➔ $\boldsymbol{t \le 2}$



Region IV

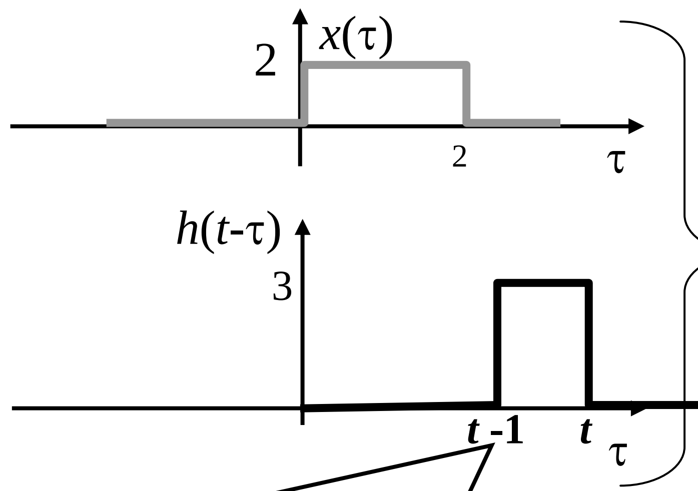
Partial τ-Overlap

$\mathbf{2 < t \le 3}$

Want $\tau_{L,t} \le 2$ ➔ t-1 ≤ 2 ➔ $\boldsymbol{t \le 3}$

Want $\tau_{R,t} > 2$ ➔ $\boldsymbol{t > 2}$

Step #5 (Continued): Find Regions of τ-Overlap



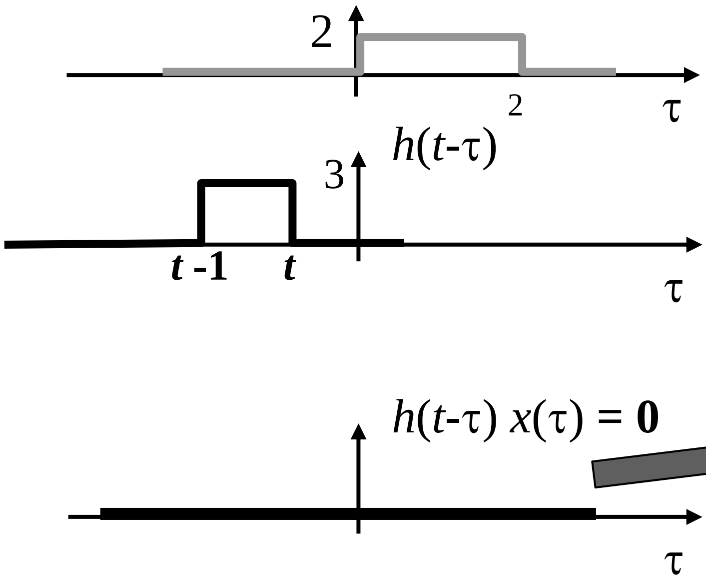
Want $\tau_{L,t} > 2$ ➔ ***t*-1 > 2** ➔ ***t* > 3**

Region V

No τ-Overlap

***t* > 3**

Step #6: Form Product & Integrate For Each Region



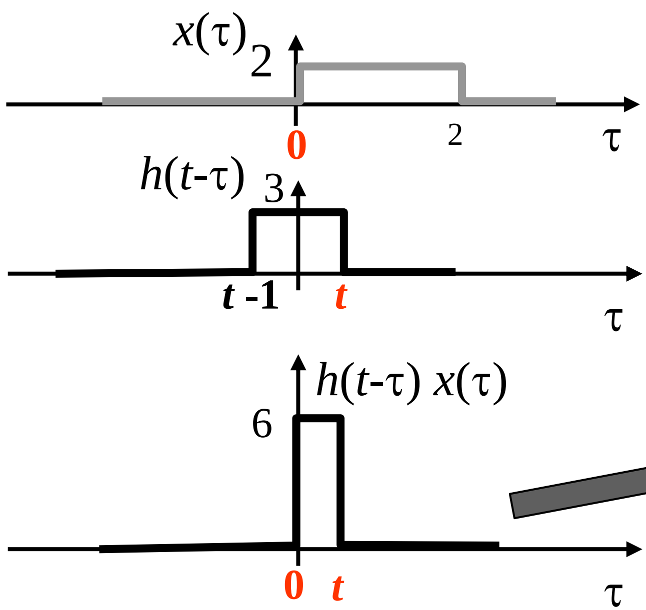
Region I: $t < 0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} 0 d\tau = 0$$

$$y(t) = 0 \quad \text{for all} \quad t < 0$$

With 0 integrand the limits don't matter!!!



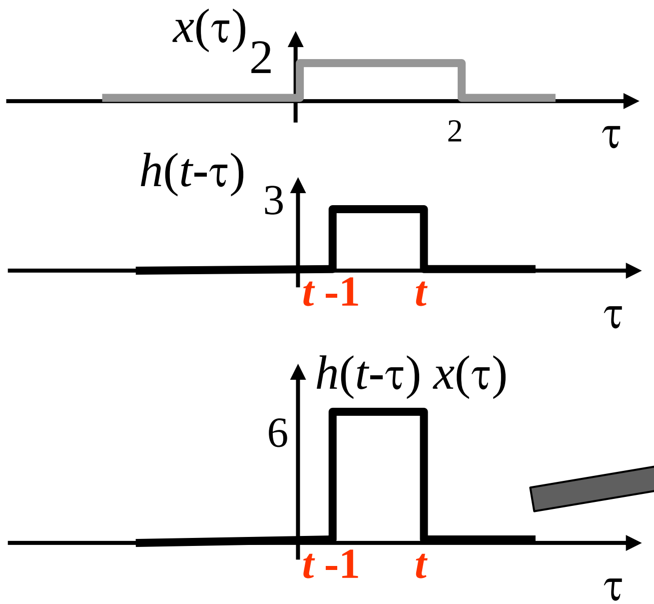
Region II: $0 \le t \le 1$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{0}^{t} 6 d\tau = [6\tau]_0^t = 6t - 6 \times 0 = 6t$$

$$y(t) = 6t \quad \text{for} \quad 0 \le t \le 1$$

Step #6 (Continued): Form Product & Integrate For Each Region

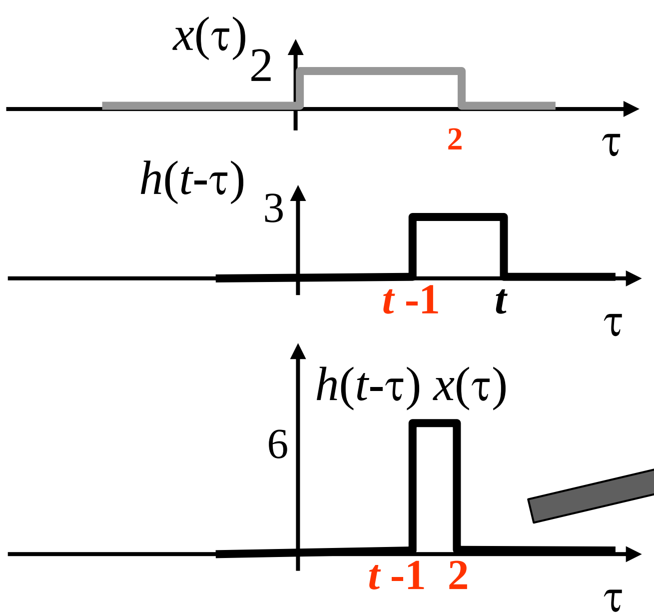


Region III: $1 < t \leq 2$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{t-1}^{t} 6d\tau = [6\tau]_{t-1}^{t} = 6t - 6(t-1) = 6$$

$y(t) = 6$ for all t such that : $1 < t \leq 2$



Region IV: $2 < t \leq 3$

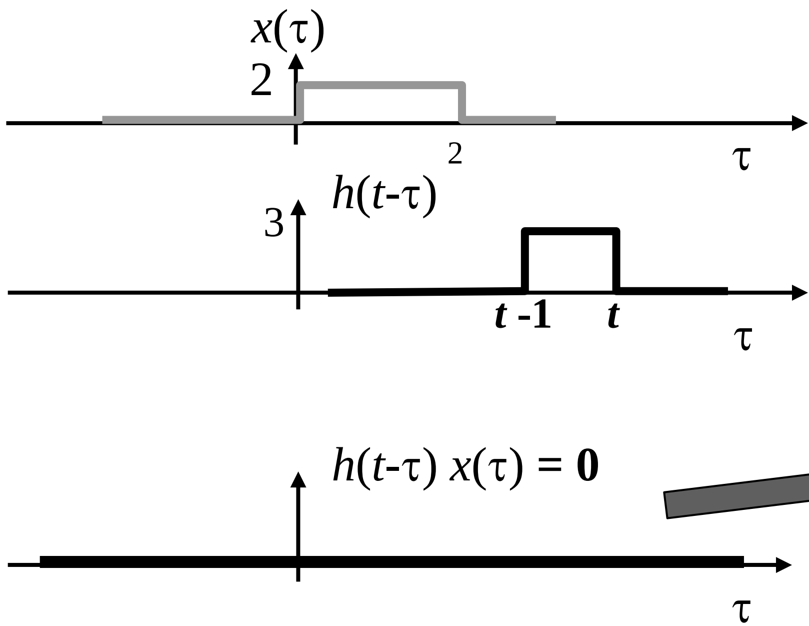
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{t-1}^{2} 6d\tau = [6\tau]_{t-1}^{2} = 6 \times 2 - 6(t-1) = -6t + 18$$

$y(t) = -6t + 18$ for $2 < t \leq 3$

Step #6 (Continued): Form Product & Integrate For Each Region

Region V: $t > 3$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} 0 d\tau = 0$$

$$y(t) = 0 \quad \text{for all} \quad t > 3$$

Step #7: "Assemble" Output Signal

Region I	Region II	Region III	Region IV	Region V
$t < 0$	$0 \le t \le 1$	$1 < t \le 2$	$2 < t \le 3$	$t > 3$
$y(t) = 0$	$y(t) = 6t$	$y(t) = 6$	$y(t) = -6t + 18$	$y(t) = 0$